

# Algebro-geometric Constructions to the Dym-type Hierarchy

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Resorting to the characteristic polynomial of Lax matrix for the Dym-type hierarchy, we define a trigonal curve, on which appropriate vector-valued Baker-Akhiezer function and meromorphic function are introduced. Based on the theory of trigonal curve and three kinds of Abelian differentials, we obtain the explicit Riemann theta function representations of the meromorphic function, from which we get the algebro-geometric constructions for the entire Dym-type hierarchy.

## 1 Introduction

As is well-known, there exist several methods to study the algebro-geometric solutions of soliton equations and from which many soliton equations associated with  $2 \times 2$  matrix spectral problems were discussed ([1, 4, 6, 7, 10, 11, 14, 16, 17, 20, 22, 26, 28, 29, 30, 31, 33, 34, 36, 39, 40] and references therein). However, the research of soliton equations associated with  $3 \times 3$  matrix spectral problems is very few, which is also much more difficult and complicated for the underlying algebraic curve is trigonal curve. It should be pointed out that this trigonal curve considerably complicates the analysis and hence makes it a rather challenging problem. More recently, according to a unified framework [5], algebro-geometric solutions for a lot of soliton hierarchies associated with  $3 \times 3$  matrix

spectral problems have been discussed, such as modified Boussinesq hierarchy [12], Kaup-Kuperschmidt hierarchy [13], three-wave resonant interaction hierarchy [19], and others [15, 38].

The Dym equation

$$u_t = (u^{-\frac{1}{2}})_{xxx}, \quad (1.1)$$

was first discovered by Harry Dym [23] and rediscovered by Li [24] and Sabatier [35]. It was shown that the Dym equation possesses many properties typical for integrable systems (see [3, 27, 37] and references therein). Moreover, the algebro-geometric solution of Dym equation was also discussed in [6, 31]. Meanwhile, the integrable extensions of Dym equation attract much attention of many researchers [2, 21, 25, 32].

By considering a  $3 \times 3$  matrix spectral problem, Prof. Geng [9] derived a hierarchy of Dym-type equations and discussed its nonlinearization. The first nontrivial member in the hierarchy is Dym-type equation

$$v_t = -\frac{1}{3}(v^{-\frac{2}{3}})_{xxxx}. \quad (1.2)$$

The principal aim of the present paper is to study algebro-geometric constructions of the Dym-type flows. With the aid of the three kinds of Abelian differentials and asymptotic expansions, we arrive at the Riemann theta function representations of the meromorphic function, and solutions for the entire Dym-type hierarchy. In this process, an explicit expression of the third kind of Abelian differential proposed by us is of great importance.

The outline of the present paper is as follows. In section 2, based on the Lenard recursion equations and the zero-curvature equation, we deduce the Dym-type hierarchy. In section 3, we define the vector-valued Baker-Akhiezer function and the associated meromorphic function, from which a trigonal curve  $\mathcal{K}_{m-1}$  of arithmetic genus  $m - 1$  is introduced with the help of the characteristic polynomial of Lax matrix for the Dym-type hierarchy. It is shown that the Dym-type hierarchy is decomposed into a system of Dubrovin-type equations. In section 4, by introducing three kinds of Abelian differentials, especially the explicit third kind, we present the Riemann theta function representations of the meromorphic function, and in particular, that of the potential  $v$  for the entire Dym-type hierarchy.

## 2 Dym-type Hierarchy

In this section, we follow the Geng [9] and derive the Dym-type hierarchy associated with the  $3 \times 3$  matrix spectral problem

$$\psi_x = U\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda v & 0 & 0 \end{pmatrix}, \quad (2.1)$$

where  $v(\neq 0)$  is a potential and  $\lambda$  is a constant spectral parameter. To this end, we introduce two sets of Lenard recursion equations

$$Kg_{j-1} = Jg_j, \quad j \geq 0, \quad (2.2)$$

$$K\hat{g}_{j-1} = J\hat{g}_j, \quad j \geq 0, \quad (2.3)$$

with two starting points

$$g_{-1} = \begin{pmatrix} v^{-\frac{2}{3}} \\ 0 \end{pmatrix}, \quad \hat{g}_{-1} = \begin{pmatrix} -\frac{1}{3}v^{-1}(v^{-\frac{1}{3}})_{xx} + \frac{1}{6}v^{-\frac{2}{3}}[(v^{-\frac{1}{3}})_x]^2 \\ v^{-\frac{1}{3}} \end{pmatrix}, \quad (2.4)$$

and two operators are defined as

$$K = \begin{pmatrix} -\frac{1}{3}\partial^5 & 0 \\ 2\partial v + v\partial & \partial^3 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -(\partial v + 2v\partial) \\ 2\partial v + v\partial & \partial^3 \end{pmatrix}.$$

It is easy to see that

$$\ker J = \{\alpha_0 g_{-1} + \beta_0 \hat{g}_{-1} \mid \forall \alpha_0, \beta_0 \in \mathbb{R}\}.$$

In order to generate a hierarchy of evolution equations associated with the spectral problem (2.1), we solve the stationary zero-curvature equation

$$V_x - [U, V] = 0, \quad V = \lambda \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}, \quad (2.5)$$

which is equivalent to

$$\begin{aligned} V_{11,x} + \lambda v V_{13} - V_{21} &= 0, \\ V_{12,x} + V_{11} - V_{22} &= 0, \\ V_{13,x} + V_{12} - V_{23} &= 0, \\ V_{21,x} + \lambda v V_{23} - V_{31} &= 0, \\ V_{22,x} + V_{21} - V_{32} &= 0, \\ V_{23,x} + V_{22} - V_{33} &= 0, \\ V_{31,x} - \lambda v (V_{11} - V_{33}) &= 0, \\ V_{32,x} - \lambda v V_{12} + V_{31} &= 0, \\ V_{33,x} - \lambda v V_{13} + V_{32} &= 0, \end{aligned} \quad (2.6)$$

where each entry  $V_{ij} = V_{ij}(a, b)$  is a Laurent expansion in  $\lambda$ :

$$\begin{aligned} V_{11} &= -\frac{1}{3}\partial^2 a - \lambda\partial b, & V_{12} &= \partial a + \lambda b, & V_{13} &= -2a, \\ V_{21} &= -\frac{1}{3}\partial^3 a - \lambda\partial^2 b - 2\lambda v a, & V_{22} &= \frac{2}{3}\partial^2 a, & V_{23} &= -\partial a + \lambda b, \\ V_{31} &= -\frac{1}{3}\partial^4 a + \lambda^2 v b, & V_{32} &= \frac{1}{3}\partial^3 a - \lambda\partial^2 b - 2\lambda v a, & V_{33} &= -\frac{1}{3}\partial^2 a + \lambda\partial b. \end{aligned} \quad (2.7)$$

Substituting (2.7) into (2.6) and expanding the functions  $a$  and  $b$  into the Laurent series in  $\lambda$

$$a = \sum_{j \geq 0} a_{j-1} \lambda^{-2j}, \quad b = \sum_{j \geq 0} b_{j-1} \lambda^{-2j}, \quad (2.8)$$

we obtain the recursion equations

$$KG_{j-1} = JG_j, \quad JG_{-1} = 0, \quad j \geq 0, \quad (2.9)$$

with  $G_j = (a_j, b_j)^T$ . Since equation  $JG_{-1} = 0$  has the general solution

$$G_{-1} = \alpha_0 g_{-1} + \beta_0 \hat{g}_{-1}, \quad (2.10)$$

$G_j$  can be expressed as

$$\begin{aligned} G_j &= \alpha_0 g_j + \beta_0 \hat{g}_j + \cdots + \alpha_j g_0 + \beta_j \hat{g}_0 \\ &\quad + \alpha_{j+1} g_{-1} + \beta_{j+1} \hat{g}_{-1}, \quad j \geq 0, \end{aligned} \quad (2.11)$$

where  $\alpha_j$  and  $\beta_j$  are arbitrary constants. Let  $\psi$  satisfy the spectral problem (2.1) and an auxiliary problem

$$\psi_{t_r} = \tilde{V}^{(r)} \psi, \quad \tilde{V}^{(r)} = \lambda \begin{pmatrix} \tilde{V}_{11}^{(r)} & \tilde{V}_{12}^{(r)} & \tilde{V}_{13}^{(r)} \\ \tilde{V}_{21}^{(r)} & \tilde{V}_{22}^{(r)} & \tilde{V}_{23}^{(r)} \\ \tilde{V}_{31}^{(r)} & \tilde{V}_{32}^{(r)} & \tilde{V}_{33}^{(r)} \end{pmatrix}, \quad (2.12)$$

where

$$\begin{aligned} \tilde{V}_{11}^{(r)} &= -\frac{1}{3}\partial^2 \tilde{a}^{(r)} - \lambda\partial \tilde{b}^{(r)}, & \tilde{V}_{12}^{(r)} &= \partial \tilde{a}^{(r)} + \lambda \tilde{b}^{(r)}, & \tilde{V}_{13}^{(r)} &= -2\tilde{a}^{(r)}, \\ \tilde{V}_{21}^{(r)} &= -\frac{1}{3}\partial^3 \tilde{a}^{(r)} - \lambda\partial^2 \tilde{b}^{(r)} - 2\lambda v \tilde{a}^{(r)}, & \tilde{V}_{22}^{(r)} &= \frac{2}{3}\partial^2 \tilde{a}^{(r)}, & \tilde{V}_{23}^{(r)} &= -\partial \tilde{a}^{(r)} + \lambda \tilde{b}^{(r)}, \\ \tilde{V}_{31}^{(r)} &= -\frac{1}{3}\partial^4 \tilde{a}^{(r)} + \lambda^2 v \tilde{b}^{(r)}, & \tilde{V}_{32}^{(r)} &= \frac{1}{3}\partial^3 \tilde{a}^{(r)} - \lambda\partial^2 \tilde{b}^{(r)} - 2\lambda v \tilde{a}^{(r)}, \\ \tilde{V}_{33}^{(r)} &= -\frac{1}{3}\partial^2 \tilde{a}^{(r)} + \lambda\partial \tilde{b}^{(r)}, & \tilde{a}^{(r)} &= \sum_{j=0}^r \tilde{a}_{j-1} \lambda^{2(r-j)}, & \tilde{b}^{(r)} &= \sum_{j=0}^r \tilde{b}_{j-1} \lambda^{2(r-j)}, \\ \tilde{G}_j &= (\tilde{a}_j, \tilde{b}_j)^T = \tilde{\alpha}_0 g_j + \tilde{\beta}_0 \hat{g}_j + \cdots + \tilde{\alpha}_j g_0 + \tilde{\beta}_j \hat{g}_0 + \tilde{\alpha}_{j+1} g_{-1} + \tilde{\beta}_{j+1} \hat{g}_{-1}, \quad j \geq -1, \end{aligned} \quad (2.13)$$

and the constants  $\tilde{\alpha}_j, \tilde{\beta}_j$  are independent of  $\alpha_j, \beta_j$ . Then the compatibility condition of (2.1) and (2.12) yields the zero-curvature equation,  $U_{t_r} - \tilde{V}_x^{(r)} + [U, \tilde{V}^{(r)}] = 0$ , which is equivalent to the hierarchy of nonlinear evolution equations

$$v_{t_r} = -\frac{1}{3}\partial^5 \tilde{a}_{r-1} = -(\partial v + 2v\partial)\tilde{b}_r. \quad (2.14)$$

The first nontrivial member in the hierarchy (2.14) is

$$v_{t_0} = -\frac{1}{3}\partial^5\{\tilde{\alpha}_0 v^{-\frac{2}{3}} + \tilde{\beta}_0[-\frac{1}{3}v^{-1}(v^{-\frac{1}{3}})_{xx} + \frac{1}{6}v^{-\frac{2}{3}}(v^{-\frac{1}{3}})_x^2]\}, \quad (2.15)$$

which is just the Dym-type equation (1.2) for  $\tilde{\alpha}_0 = 1, \tilde{\beta}_0 = 0, t_0 = t$ .

### 3 The Baker-Akhiezer Function

In this section, we shall introduce the vector-valued Baker-Akhiezer function, meromorphic function and trigonal curve associated with the Dym-type hierarchy. Then we derive a system of Dubrovin-type differential equations.

We introduce the vector-valued Baker-Akhiezer function

$$\begin{aligned} \psi_x(P, x, x_0, t_r, t_{0,r}) &= U(v, \lambda)\psi(P, x, x_0, t_r, t_{0,r}), \\ \psi_{t_r}(P, x, x_0, t, t_{0,r}) &= \tilde{V}^{(r)}(v, \lambda)\psi(P, x, x_0, t_r, t_{0,r}), \\ \lambda^{-1}V^{(n)}(v, \lambda)\psi(P, x, x_0, t, t_{0,r}) &= y(P)\psi(P, x, x_0, t_r, t_{0,r}), \\ \psi_1(P, x_0, x_0, t_{0,r}, t_{0,r}) &= 1, \quad P = (\lambda, y), \quad x, t_r \in \mathbb{C}. \end{aligned} \quad (3.1)$$

Here  $V^{(n)} = \lambda \left( V_{ij}^{(n)} \right)_{3 \times 3}$  and

$$\begin{aligned} V_{11}^{(n)} &= -\frac{1}{3}\partial^2 a^{(n)} - \lambda \partial b^{(n)}, \quad V_{12}^{(n)} = \partial a^{(n)} + \lambda b^{(n)}, \quad V_{13}^{(n)} = -2a^{(n)}, \\ V_{21}^{(n)} &= -\frac{1}{3}\partial^3 a^{(n)} - \lambda \partial^2 b^{(n)} - 2\lambda v a^{(n)}, \quad V_{22}^{(n)} = \frac{2}{3}\partial^2 a^{(n)}, \quad V_{23}^{(n)} = -\partial a^{(n)} + \lambda b^{(n)}, \\ V_{31}^{(n)} &= -\frac{1}{3}\partial^4 a^{(n)} + \lambda^2 v b^{(n)}, \quad V_{32}^{(n)} = \frac{1}{3}\partial^3 a^{(n)} - \lambda \partial^2 b^{(n)} - 2\lambda v a^{(n)}, \\ V_{33}^{(n)} &= -\frac{1}{3}\partial^2 a^{(n)} + \lambda \partial b^{(n)}, \quad a^{(n)} = \sum_{j=0}^n a_{j-1} \lambda^{2(n-j)}, \quad b^{(n)} = \sum_{j=0}^n b_{j-1} \lambda^{2(n-j)}, \end{aligned}$$

in which  $a_j, b_j$  are determined by (2.11). The compatibility conditions of the first three equations in (3.1) yield that

$$U_{t_r} - \tilde{V}_x^{(r)} + [U, \tilde{V}^{(r)}] = 0, \quad (3.2)$$

$$-V_x^{(n)} + [U, V^{(n)}] = 0, \quad (3.3)$$

$$-V_{t_r}^{(n)} + [\tilde{V}^{(r)}, V^{(n)}] = 0. \quad (3.4)$$

A direct calculation shows that  $yI - \lambda^{-1}V^{(n)}$  satisfies (3.3) and (3.4). Hence the characteristic polynomial of Lax matrix  $\lambda^{-1}V^{(n)}$  for the Dym-type hierarchy is a constant independent of variables  $x$  and  $t_r$ , and possesses following expansion

$$\mathcal{F}_m(\lambda, y) = \det(yI - \lambda^{-1}V^{(n)}) = y^3 + yS_m(\lambda) - T_m(\lambda), \quad (3.5)$$

where  $S_m(\lambda)$  and  $T_m(\lambda)$  are polynomials of  $\lambda$  with constant coefficients

$$S_m(\lambda) = \begin{vmatrix} V_{11}^{(n)} & V_{12}^{(n)} \\ V_{21}^{(n)} & V_{22}^{(n)} \end{vmatrix} + \begin{vmatrix} V_{11}^{(n)} & V_{13}^{(n)} \\ V_{31}^{(n)} & V_{33}^{(n)} \end{vmatrix} + \begin{vmatrix} V_{22}^{(n)} & V_{23}^{(n)} \\ V_{32}^{(n)} & V_{33}^{(n)} \end{vmatrix}, \quad (3.6)$$

$$T_m(\lambda) = \begin{vmatrix} V_{11}^{(n)} & V_{12}^{(n)} & V_{13}^{(n)} \\ V_{21}^{(n)} & V_{22}^{(n)} & V_{23}^{(n)} \\ V_{31}^{(n)} & V_{32}^{(n)} & V_{33}^{(n)} \end{vmatrix} = \begin{cases} \beta_0^3 \lambda^{6n+4} + \dots, & \beta_0 \neq 0, \alpha_0 \in \mathbb{R}, \\ -8\alpha_0^3 \lambda^{6n+2} + \dots, & \beta_0 = 0, \alpha_0 \neq 0. \end{cases}$$

It is evident that  $T_m(\lambda)$  is a polynomial of degree  $6n + 4 = 3(2n + 1) + 1$  and  $6n + 2 = 3(2n) + 2$  as  $\beta_0 \neq 0, \alpha_0 \in \mathbb{R}$  and  $\beta_0 = 0, \alpha_0 \neq 0$ , respectively. Then  $\mathcal{F}_m(\lambda, y) = 0$  naturally leads to a trigonal curve

$$\mathcal{K}_{m-1}: \quad \mathcal{F}_m(\lambda, y) = y^3 + yS_m(\lambda) - T_m(\lambda) = 0, \quad (3.7)$$

with  $m = 6n + 4$  or  $m = 6n + 2$ .

For the convenience, we denote the compactification of the curve  $\mathcal{K}_{m-1}$  by the same symbol  $\mathcal{K}_{m-1}$ . Thus  $\mathcal{K}_{m-1}$  becomes a three-sheeted Riemann surface of arithmetic genus  $m - 1$  if it is nonsingular or smooth, which means that  $(\frac{\partial \mathcal{F}_m(\lambda, y)}{\partial \lambda}, \frac{\partial \mathcal{F}_m(\lambda, y)}{\partial y})|_{(\lambda, y)=(\lambda', y')} \neq (0, 0)$  at each point  $P' = (\lambda', y') \in \mathcal{K}_{m-1}$ .

A meromorphic function  $\phi(P, x, t_r)$  on  $\mathcal{K}_{m-1}$  is defined as

$$\phi(P) = \phi(P, x, t_r) = v^{-\frac{1}{3}} \frac{\psi_{1,x}(P, x, x_0, t_r, t_{0,r})}{\psi_1(P, x, x_0, t_r, t_{0,r})} = v^{-\frac{1}{3}} \frac{\psi_2(P, x, x_0, t_r, t_{0,r})}{\psi_1(P, x, x_0, t_r, t_{0,r})}, \quad P \in \mathcal{K}_{m-1}. \quad (3.8)$$

It infers from (3.1) and (3.8) that

$$\phi(P) = v^{-\frac{1}{3}} \frac{yV_{23}^{(n)} + C_m}{yV_{13}^{(n)} + A_m} = \frac{v^{-\frac{1}{3}} F_m}{y^2 V_{23}^{(n)} - yC_m + D_m} = \frac{y^2 V_{13}^{(n)} - yA_m + B_m}{v^{\frac{1}{3}} E_{m-1}}, \quad (3.9)$$

where

$$\begin{aligned} A_m &= V_{12}^{(n)} V_{23}^{(n)} - V_{13}^{(n)} V_{22}^{(n)}, \quad C_m = V_{13}^{(n)} V_{21}^{(n)} - V_{11}^{(n)} V_{23}^{(n)}, \\ B_m &= V_{13}^{(n)} (V_{11}^{(n)} V_{33}^{(n)} - V_{13}^{(n)} V_{31}^{(n)}) + V_{12}^{(n)} (V_{11}^{(n)} V_{23}^{(n)} - V_{13}^{(n)} V_{21}^{(n)}), \\ D_m &= V_{23}^{(n)} (V_{22}^{(n)} V_{33}^{(n)} - V_{23}^{(n)} V_{32}^{(n)}) + V_{21}^{(n)} (V_{13}^{(n)} V_{22}^{(n)} - V_{12}^{(n)} V_{23}^{(n)}), \\ E_{m-1} &= (V_{13}^{(n)})^2 V_{32}^{(n)} + V_{12}^{(n)} V_{13}^{(n)} (V_{22}^{(n)} - V_{33}^{(n)}) - (V_{12}^{(n)})^2 V_{23}^{(n)}, \\ F_m &= (V_{23}^{(n)})^2 V_{31}^{(n)} + V_{21}^{(n)} V_{23}^{(n)} (V_{11}^{(n)} - V_{33}^{(n)}) - V_{13}^{(n)} (V_{21}^{(n)})^2. \end{aligned} \quad (3.10)$$

Taking (3.7) and (3.9) into account, we arrive at some important identities among polynomials  $A_m, B_m, C_m, D_m, E_{m-1}, F_m, S_m, T_m$ :

$$\begin{aligned} V_{13}^{(n)} F_m &= V_{23}^{(n)} D_m - (V_{23}^{(n)})^2 S_m - C_m^2, \\ A_m F_m &= (V_{23}^{(n)})^2 T_m + C_m D_m, \end{aligned} \quad (3.11)$$

$$\begin{aligned} V_{23}^{(n)} E_{m-1} &= V_{13}^{(n)} B_m - (V_{13}^{(n)})^2 S_m - A_m^2, \\ C_m E_{m-1} &= (V_{13}^{(n)})^2 T_m + A_m B_m, \end{aligned} \quad (3.12)$$

$$\begin{aligned} V_{23}^{(n)} B_m + V_{13}^{(n)} D_m - V_{13}^{(n)} V_{23}^{(n)} S_m + A_m C_m &= 0, \\ V_{13}^{(n)} V_{23}^{(n)} T_m + V_{23}^{(n)} A_m S_m + V_{13}^{(n)} C_m S_m - B_m C_m - A_m D_m &= 0, \\ V_{23}^{(n)} A_m T_m + V_{13}^{(n)} C_m T_m - B_m D_m + E_{m-1} F_m &= 0, \end{aligned} \quad (3.13)$$

$$\begin{aligned} E_{m-1,x} &= -2V_{13}^{(n)} S_m + 3B_m, \\ V_{23}^{(n)} F_{m,x} &= -3V_{22}^{(n)} F_m + V_{21}^{(n)} (2V_{23}^{(n)} S_m - 3D_m). \end{aligned} \quad (3.14)$$

Now, we define the holomorphic mapping  $*$ , changing sheets, by

$$\begin{aligned} * : \begin{cases} \mathcal{K}_{m-1} \rightarrow \mathcal{K}_{m-1} \\ P = (\lambda, y_i(\lambda)) \rightarrow P^* = (\lambda, y_{i+1(\text{mod}3)}(\lambda)), \quad i = 0, 1, 2 \end{cases}, \\ P^{**} := (P^*)^*, \quad \text{etc.}, \end{aligned} \quad (3.15)$$

where  $y_i(\lambda), i = 0, 1, 2$ , denote the three branches of  $y(P)$  satisfying  $\mathcal{F}_m(\lambda, y) = 0$ , namely,

$$(y - y_0(\lambda))(y - y_1(\lambda))(y - y_2(\lambda)) = y^3 + yS_m(\lambda) - T_m(\lambda) = 0. \quad (3.16)$$

Consequently, we have

$$\begin{aligned} y_0 + y_1 + y_2 &= 0, \quad y_0 y_1 + y_0 y_2 + y_1 y_2 = S_m(\lambda), \\ y_0 y_1 y_2 &= T_m(\lambda), \quad y_0^2 + y_1^2 + y_2^2 = -2S_m(\lambda), \\ y_0^3 + y_1^3 + y_2^3 &= 3T_m(\lambda), \quad y_0^2 y_1^2 + y_0^2 y_2^2 + y_1^2 y_2^2 = S_m^2(\lambda). \end{aligned} \quad (3.17)$$

In what follows, we shall summarize some properties of the meromorphic function  $\phi(P, x, t_r)$  without proofs.

$$[v^{\frac{1}{3}}\phi(P)]_{xx} + 3v^{\frac{1}{3}}\phi(P)[v^{\frac{1}{3}}\phi(P)]_x + v\phi^3(P) = \lambda v, \quad (3.18)$$

$$[v^{\frac{1}{3}}\phi(P)]_{t_r} = \lambda \left( \tilde{V}_{11}^{(r)} + v^{\frac{1}{3}}\tilde{V}_{12}^{(r)}\phi(P) + \tilde{V}_{13}^{(r)}[(v^{\frac{1}{3}}\phi(P))_x + v^{\frac{2}{3}}\phi^2(P)] \right)_x, \quad (3.19)$$

$$v\phi(P)\phi(P^*)\phi(P^{**}) = -\frac{F_m}{E_{m-1}}, \quad (3.20)$$

$$v^{\frac{1}{3}}[\phi(P) + \phi(P^*) + \phi(P^{**})] = \frac{E_{m-1,x}}{E_{m-1}}, \quad (3.21)$$

$$\frac{1}{\phi(P)} + \frac{1}{\phi(P^*)} + \frac{1}{\phi(P^{**})} = v^{\frac{1}{3}} \left[ -3\frac{V_{22}^{(n)}}{V_{21}^{(n)}} - \frac{V_{23}^{(n)} F_{m,x}}{V_{21}^{(n)} F_m} \right], \quad (3.22)$$

$$\begin{aligned} [v^{\frac{1}{3}}\phi(P) + v^{\frac{1}{3}}\phi(P^*) + v^{\frac{1}{3}}\phi(P^{**})]_x + v^{\frac{2}{3}}[\phi^2(P) + \phi^2(P^*) \\ + \phi^2(P^{**})] = -3\frac{V_{11}^{(n)}}{V_{13}^{(n)}} - \frac{V_{12}^{(n)} E_{m-1,x}}{V_{13}^{(n)} E_{m-1}}. \end{aligned} \quad (3.23)$$

**Lemma 3.1.** Assume that (3.1) and (3.2) hold, and let  $(\lambda, x, t_r) \in \mathbb{C}^3$ . Then

$$E_{m-1,t_r} = \lambda E_{m-1,x} [\tilde{V}_{12}^{(r)} - \frac{\tilde{V}_{13}^{(r)}}{V_{13}^{(n)}} V_{12}^{(n)}] + 3\lambda E_{m-1} [\tilde{V}_{11}^{(r)} - \frac{\tilde{V}_{13}^{(r)}}{V_{13}^{(n)}} V_{11}^{(n)}], \quad (3.24)$$

$$F_{m,t_r} = \lambda F_{m,x} [\tilde{V}_{23}^{(r)} - \frac{\tilde{V}_{21}^{(r)}}{V_{21}^{(n)}} V_{23}^{(n)}] + 3\lambda F_m [\tilde{V}_{22}^{(r)} - \frac{\tilde{V}_{21}^{(r)}}{V_{21}^{(n)}} V_{22}^{(n)}]. \quad (3.25)$$

**Proof.** Differentiating (3.21) with respect to  $t_r$  and using (3.19), (3.21) and (3.23), we have

$$\begin{aligned} \partial_x \partial_{t_r} (\ln E_{m-1}) &= [v^{\frac{1}{3}} \phi(P) + v^{\frac{1}{3}} \phi(P^*) + v^{\frac{1}{3}} \phi(P^{**})]_{t_r} \\ &= \lambda \left( 3\tilde{V}_{11}^{(r)} + v^{\frac{1}{3}} \tilde{V}_{12}^{(r)} [\phi(P) + \phi(P^*) + \phi(P^{**})] \right. \\ &\quad \left. + \tilde{V}_{13}^{(r)} [(v^{\frac{1}{3}} \phi(P) + v^{\frac{1}{3}} \phi(P^*) + v^{\frac{1}{3}} \phi(P^{**}))_x] \right. \\ &\quad \left. + v^{\frac{2}{3}} (\phi^2(P) + \phi^2(P^*) + \phi^2(P^{**})) \right)_x \\ &= \lambda \left[ (\tilde{V}_{12}^{(r)} - \frac{\tilde{V}_{13}^{(r)}}{V_{13}^{(n)}} V_{12}^{(n)}) \frac{E_{m-1,x}}{E_{m-1}} + 3(\tilde{V}_{11}^{(r)} - \frac{\tilde{V}_{13}^{(r)}}{V_{13}^{(n)}} V_{11}^{(n)}) \right]_x. \end{aligned}$$

Integrating the above equation with respect to  $x$  and choosing the integration constant as zero imply the first equation in (3.24). Differentiating (3.20) with respect to  $t_r$ , an analogous process shows (3.25).

By observing (2.11) and (3.10), we can easily find that  $E_{m-1}$  and  $F_m$  are polynomials with respect to  $\lambda$  of degree  $m-1$  and  $m$ , respectively. Therefore

$$E_{m-1}(\lambda, x, t_r) = -\epsilon(m) v^{-1} \prod_{j=1}^{m-1} (\lambda - \mu_j(x, t_r)), \quad (3.26)$$

$$F_m(\lambda, x, t_r) = \epsilon(m) \prod_{l=0}^{m-1} (\lambda - \nu_l(x, t_r)), \quad (3.27)$$

with

$$\epsilon(m) = \begin{cases} \beta_0^3, & m = 6n + 4, \\ 8\alpha_0^3, & m = 6n + 2. \end{cases}$$

Let us denote

$$\begin{aligned} \hat{\mu}_j(x, t_r) &= (\mu_j(x, t_r), y(\hat{\mu}_j(x, t_r))) \\ &= \left( \mu_j(x, t_r), -\frac{A_m(\mu_j(x, t_r), x, t_r)}{V_{13}^{(n)}(\mu_j(x, t_r), x, t_r)} \right) \in \mathcal{K}_{m-1}, \quad 1 \leq j \leq m-1, \end{aligned} \quad (3.28)$$

$$\begin{aligned} \hat{\nu}_l(x, t_r) &= (\nu_l(x, t_r), y(\hat{\nu}_l(x, t_r))) \\ &= \left( \nu_l(x, t_r), -\frac{C_m(\nu_l(x, t_r), x, t_r)}{V_{23}^{(n)}(\nu_l(x, t_r), x, t_r)} \right) \in \mathcal{K}_{m-1}, \quad 0 \leq l \leq m-1, \end{aligned} \quad (3.29)$$



then it is easy to see that the following Lemma holds.

**Lemma 3.2.** Suppose the zeros  $\{\mu_j(x, t_r)\}_{j=1}^{m-1}$  and  $\{\nu_l(x, t_r)\}_{l=0}^{m-1}$  of  $E_{m-1}(\lambda, x, t_r)$  and  $F_m(\lambda, x, t_r)$  remain distinct for  $(x, t_r) \in \Omega_\mu$  and  $(x, t_r) \in \Omega_\nu$ , respectively, where  $\Omega_\mu, \Omega_\nu \subseteq \mathbb{C}^2$  are open and connected. Then  $\{\mu_j(x, t_r)\}_{j=1}^{m-1}$  and  $\{\nu_l(x, t_r)\}_{l=0}^{m-1}$  satisfy the Dubrovin-type equations

$$\mu_{j,x} = \frac{vV_{13}^{(n)}(\mu_j, x, t_r)[3y^2(\hat{\mu}_j) + S_m(\mu_j)]}{\epsilon(m) \prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\mu_j - \mu_k)}, \quad 1 \leq j \leq m-1, \quad (3.30)$$

$$\begin{aligned} \mu_{j,t_r} &= v\mu_j[V_{13}^{(n)}(\lambda, x, t_r)\tilde{V}_{12}^{(r)}(\lambda, x, t_r) - \tilde{V}_{13}^{(r)}(\lambda, x, t_r)V_{12}^{(n)}(\lambda, x, t_r)]_{\lambda=\mu_j} \\ &\times \frac{[3y^2(\hat{\mu}_j) + S_m(\mu_j)]}{\epsilon(m) \prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\mu_j - \mu_k)}, \quad 1 \leq j \leq m-1, \end{aligned} \quad (3.31)$$

$$\nu_{l,x} = \frac{V_{21}^{(n)}(\nu_l, x, t_r)[3y^2(\hat{\nu}_l) + S_m(\nu_l)]}{\epsilon(m) \prod_{\substack{k=0 \\ k \neq l}}^{m-1} (\nu_l - \nu_k)}, \quad 0 \leq l \leq m-1, \quad (3.32)$$

$$\begin{aligned} \nu_{l,t_r} &= \nu_l[V_{21}^{(n)}(\lambda, x, t_r)\tilde{V}_{23}^{(r)}(\lambda, x, t_r) - \tilde{V}_{21}^{(r)}(\lambda, x, t_r)V_{23}^{(n)}(\lambda, x, t_r)]_{\lambda=\nu_l} \\ &\times \frac{[3y^2(\hat{\nu}_l) + S_m(\nu_l)]}{\epsilon(m) \prod_{\substack{k=0 \\ k \neq l}}^{m-1} (\nu_l - \nu_k)}, \quad 0 \leq l \leq m-1. \end{aligned} \quad (3.33)$$

**Proof.** We just need to prove (3.30) for the proofs of (3.31)-(3.33) are similar to (3.30). Substituting  $\lambda = \mu_j$  into the first expression in (3.14), and using (3.12) and (3.29), we get

$$E_{m-1,x}(\mu_j, x, t_r) = V_{13}^{(n)}(\mu_j, x, t_r)[3y^2(\hat{\mu}_j) + S_m(\mu_j)]. \quad (3.34)$$

On the other hand, differentiating (3.26) with respect to  $x$  and inserting  $\lambda = \mu_j$  into it give rise to

$$E_{m-1,x}(\mu_j, x, t_r) = \epsilon(m)v^{-1}\mu_{j,x} \prod_{\substack{k=1 \\ k \neq j}}^{m-1} (\mu_j - \mu_k). \quad (3.35)$$

A comparison of (3.34) and (3.35) yields (3.30).

## 4 Algebro-geometric Constructions to the Dym-type Hierarchy

In this section, we shall derive explicit Riemann theta function representations for the meromorphic function  $\phi(P, x, t_r)$ , and in particular, that of potential  $v(x, t_r)$  for the entire Dym-type hierarchy.

Taking the local coordinate  $\zeta = \lambda^{-\frac{1}{3}}$  near  $P_\infty \in \mathcal{K}_{m-1}$  in (3.18), the Laurent series of  $\phi(P, x, t_r)$  can be explicitly expressed as

$$\phi(P, x, t_r) \underset{\zeta \rightarrow 0}{=} \frac{1}{\zeta} \sum_{j=0}^{\infty} \kappa_j(x, t_r) \zeta^j, \quad P \rightarrow P_\infty, \quad (4.1)$$

where

$$\begin{aligned} \kappa_0 &= 1, \quad \kappa_1 = (v^{-\frac{1}{3}})_x, \quad \kappa_2 = \frac{2}{9}v^{-\frac{5}{3}}v_{xx} - \frac{7}{27}v^{-\frac{8}{3}}v_x^2, \quad \kappa_3 = \frac{1}{3}v^{-\frac{2}{3}}(v^{-\frac{1}{3}})_{xxx}, \\ \kappa_j &= -\frac{1}{3}[v^{-1}(v^{\frac{1}{3}}\kappa_{j-2})_{xx} + 3v^{-\frac{2}{3}}\sum_{i=0}^{j-1}\kappa_{j-1-i}(v^{\frac{1}{3}}\kappa_i)_x + \sum_{i=1}^{j-1}\kappa_i\kappa_{j-i} \\ &\quad + \sum_{i=1}^{j-1}\sum_{l=0}^{j-i}\kappa_i\kappa_l\kappa_{j-i-l}], \quad (j \geq 2). \end{aligned} \quad (4.2)$$

Defining the positive divisors on  $\mathcal{K}_{m-1}$  of degree  $m-1$

$$\mathcal{D}_{P_1, \dots, P_{m-1}} : \begin{cases} \mathcal{K}_{m-1} \rightarrow \mathbb{N}_0, \\ P \rightarrow \mathcal{D}_{P_1, \dots, P_{m-1}}(P) = \begin{cases} k, & \text{if } P \text{ occurs } k \text{ times in } \{P_1, \dots, P_{m-1}\} \\ 0, & \text{if } P \notin \{P_1, \dots, P_{m-1}\} \end{cases} \end{cases}$$

with  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , one obtains from (3.9) and (4.1) that the divisor  $(\phi(P, x, t_r))$  of  $\phi(P, x, t_r)$  is given by

$$(\phi(P, x, t_r)) = \mathcal{D}_{\hat{\nu}_0(x, t_r), \hat{\nu}_1(x, t_r), \dots, \hat{\nu}_{m-1}(x, t_r)}(P) - \mathcal{D}_{P_\infty, \hat{\mu}_1(x, t_r), \dots, \hat{\mu}_{m-1}(x, t_r)}(P), \quad (4.3)$$

which implies that  $\hat{\nu}_0(x, t_r), \hat{\nu}_1(x, t_r), \dots, \hat{\nu}_{m-1}(x, t_r)$  are  $m$  zeros and  $P_\infty, \hat{\mu}_1(x, t_r), \dots, \hat{\mu}_{m-1}(x, t_r)$  are  $m$  poles of  $\phi(P, x, t_r)$ .

Equip the Riemann surface  $\mathcal{K}_{m-1}$  with canonical basis of cycles  $\{\mathbf{a}_j, \mathbf{b}_j\}_{j=1}^{m-1}$ , which admits intersection numbers

$$\mathbf{a}_j \circ \mathbf{b}_k = \delta_{j,k}, \quad \mathbf{a}_j \circ \mathbf{a}_k = 0, \quad \mathbf{b}_j \circ \mathbf{b}_k = 0, \quad j, k = 1, \dots, m-1,$$

and the basis of holomorphic differentials

$$\begin{aligned} \tilde{\omega}_l(P) &= \frac{1}{3y^2(P) + S_m(\lambda)} \begin{cases} \lambda^{l-1}d\lambda, & 1 \leq l \leq m-2n-2, \\ y(P)\lambda^{l+2n-m+1}d\lambda, & m-2n-1 \leq l \leq m-1, \end{cases} \quad m = 6n+4, \\ \tilde{\omega}_l(P) &= \frac{1}{3y^2(P) + S_m(\lambda)} \begin{cases} \lambda^{l-1}d\lambda, & 1 \leq l \leq m-2n-1, \\ y(P)\lambda^{l+2n-m}d\lambda, & m-2n \leq l \leq m-1, \end{cases} \quad m = 6n+2. \end{aligned} \quad (4.4)$$

Thus the period matrices  $A$  and  $B$  constructed by

$$A_{jk} = \int_{\mathfrak{a}_k} \tilde{\omega}_j, \quad B_{jk} = \int_{\mathfrak{b}_k} \tilde{\omega}_j, \quad (4.5)$$

are invertible. Defining the matrix  $C = A^{-1}$ ,  $\tau = CB$ , the Riemannian bilinear relation makes it possible to verify that the matrix  $\tau$  is symmetric ( $\tau_{jk} = \tau_{kj}$ ) and has positive definite imaginary part ( $\text{Im } \tau > 0$ ) ([8, 18]). If we normalize  $\tilde{\omega} = (\tilde{\omega}_1, \dots, \tilde{\omega}_{m-1})$  into new basis  $\omega = (\omega_1, \dots, \omega_{m-1})$

$$\omega_j = \sum_{l=1}^{m-1} C_{jl} \tilde{\omega}_l, \quad (4.6)$$

then we have

$$\int_{\mathfrak{a}_k} \omega_j = \delta_{jk}, \quad \int_{\mathfrak{b}_k} \omega_j = \tau_{jk}, \quad j, k = 1, \dots, m-1.$$

A straightforward calculation yields the following asymptotic expansions:

$$y(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \zeta^{-6n-4}[\beta_0 - 2\alpha_0\zeta^2 + O(\zeta^4)], & P \rightarrow P_\infty, \quad m = 6n + 4, \\ -\zeta^{-6n-2}[2\alpha_0 + O(\zeta^4)], & P \rightarrow P_\infty, \quad m = 6n + 2, \end{cases} \quad (4.7)$$

$$S_m(\lambda) \underset{\zeta \rightarrow 0}{=} \begin{cases} 6\alpha_0\beta_0\zeta^{-12n-6}[1 + O(\zeta^6)], & P \rightarrow P_\infty, \quad m = 6n + 4, \\ 6\alpha_0\beta_1\zeta^{-12n}[1 + O(\zeta^6)], & P \rightarrow P_\infty, \quad m = 6n + 2, \end{cases} \quad (4.8)$$

$$\omega_j \underset{\zeta \rightarrow 0}{=} \begin{cases} (-\frac{C_{j,m-1}}{\beta_0} - \frac{C_{j,4n+2}}{\beta_0^2}\zeta + O(\zeta^3))d\zeta, & P \rightarrow P_\infty, \quad m = 6n + 4, \\ (-\frac{C_{j,4n+1}}{4\alpha_0^2} + \frac{C_{j,m-1}}{2\alpha_0}\zeta + O(\zeta^3))d\zeta, & P \rightarrow P_\infty, \quad m = 6n + 2, \end{cases} \quad (4.9)$$

$j = 1, \dots, m-1.$

Let  $\omega_{P_\infty,2}^{(2)}(P)$  denote the normalized Abelian differential of the second kind, which is holomorphic on  $\mathcal{K}_{m-1} \setminus \{P_\infty\}$  with a pole of order 2 at  $P_\infty$  and satisfies

$$\int_{\mathfrak{a}_j} \omega_{P_\infty,2}^{(2)}(P) = 0, \quad j = 1, \dots, m-1. \quad (4.10)$$

$$\omega_{P_\infty,2}^{(2)}(P) = (\zeta^{-2} + O(1))d\zeta, \quad P \rightarrow P_\infty. \quad (4.11)$$

The  $b$ -periods of the differential  $\omega_{P_\infty,2}^{(2)}$  are denoted by

$$U_2^{(2)} = (U_{2,1}^{(2)}, \dots, U_{2,m-1}^{(2)}),$$

$$U_{2,j}^{(2)} = \frac{1}{2\pi i} \int_{\mathfrak{b}_j} \omega_{P_\infty,2}^{(2)}(P) = \begin{cases} -\frac{C_{j,m-1}}{\beta_0}, & m = 6n + 4, \\ -\frac{C_{j,4n+1}}{4\alpha_0^2}, & m = 6n + 2, \end{cases} \quad j = 1, \dots, m-1. \quad (4.12)$$

Furthermore, let  $\omega_{P_\infty, \hat{\nu}_0(x, t_r)}^{(3)}(P)$  denote the normalized Abelian differential of the third kind defined by

$$\omega_{P_\infty, \hat{\nu}_0(x, t_r)}^{(3)}(P) = -\frac{y^2(P) + 2y^2(\hat{\nu}_0(x, t_r)) + S_m(\nu_0(x, t_r))}{\lambda - \nu_0(x, t_r)} \frac{d\lambda}{3y^2(P) + S_m(\lambda)} + \sum_{j=1}^{m-1} \gamma_j \omega_j, \quad (4.13)$$

which is holomorphic on  $\mathcal{K}_{m-1} \setminus \{P_\infty, \hat{\nu}_0(x, t_r)\}$  and has simple poles at  $P_\infty$  and  $\hat{\nu}_0(x, t_r)$  with corresponding residues  $+1$  and  $-1$ . The constants  $\{\gamma_j\}_{j=1}^{m-1}$  are determined by the normalization condition

$$\int_{\mathfrak{a}_j} \omega_{P_\infty, \hat{\nu}_0(x, t_r)}^{(3)}(P) = 0, \quad j = 1, \dots, m-1. \quad (4.14)$$

A direct calculation shows

$$\omega_{P_\infty, \hat{\nu}_0(x, t_r)}^{(3)}(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} (\zeta^{-1} - \delta(m) + O(\zeta))d\zeta, & P \rightarrow P_\infty, \\ (-\zeta^{-1} + O(1))d\zeta, & P \rightarrow \hat{\nu}_0(x, t_r), \end{cases} \quad (4.15)$$

with

$$\delta(m) = \begin{cases} \frac{1}{\beta_0} \sum_{j=1}^{m-1} \gamma_j C_{j, m-1}, & m = 6n + 4, \\ \frac{1}{4\alpha_0^2} \sum_{j=1}^{m-1} \gamma_j C_{j, 4n+1}, & m = 6n + 2. \end{cases} \quad (4.16)$$

Then

$$\int_{Q_0}^P \omega_{P_\infty, \hat{\nu}_0(x, t_r)}^{(3)}(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \ln \zeta + e_\infty^{(3)}(Q_0) - \delta(m)\zeta + O(\zeta^2), & P \rightarrow P_\infty, \\ -\ln \zeta + e_0^{(3)}(Q_0) + O(\zeta), & P \rightarrow \hat{\nu}_0(x, t_r), \end{cases} \quad (4.17)$$

with  $Q_0$  a chosen base point on  $\mathcal{K}_{m-1} \setminus \{P_\infty, \hat{\nu}_0(x, t_r)\}$  and  $e_\infty^{(3)}(Q_0), e_0^{(3)}(Q_0)$  two integration constants.

Let  $\mathcal{T}_{m-1} = \{\underline{N} + \tau \underline{L}, \underline{N}, \underline{L} \in \mathbb{Z}^{m-1}\}$  be a period lattice. The complex torus  $\mathcal{J}_{m-1} = \mathbb{C}^{m-1} / \mathcal{T}_{m-1}$  is called a Jacobian variety of  $\mathcal{K}_{m-1}$ . The Abelian mapping  $\mathcal{A} : \mathcal{K}_{m-1} \rightarrow \mathcal{J}_{m-1}$  is defined as

$$\mathcal{A}(P) = \left( \mathcal{A}_1(P), \dots, \mathcal{A}_{m-1}(P) \right) = \left( \int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_{m-1} \right) \pmod{\mathcal{T}_{m-1}},$$

and is extended linearly to the divisor group  $\text{Div}(\mathcal{K}_{m-1})$

$$\mathcal{A}\left(\sum_k n_k P_k\right) = \sum_k n_k \mathcal{A}(P_k),$$

which enables us to give the Abel-Jacobi coordinates for the nonspecial divisor  $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)} = \sum_{k=1}^{m-1} \hat{\mu}_k(x, t_r)$  and  $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)} = \sum_{k=1}^{m-1} \hat{\nu}_k(x, t_r)$ :

$$\begin{aligned} \rho^{(1)}(x, t_r) &= \mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \mathcal{A}\left(\sum_{k=1}^{m-1} \hat{\mu}_k(x, t_r)\right) = \sum_{k=1}^{m-1} \mathcal{A}(\hat{\mu}_k(x, t_r)) = \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\mu}_k(x,t_r)} \omega, \\ \rho^{(2)}(x, t_r) &= \mathcal{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) = \mathcal{A}\left(\sum_{k=1}^{m-1} \hat{\nu}_k(x, t_r)\right) = \sum_{k=1}^{m-1} \mathcal{A}(\hat{\nu}_k(x, t_r)) = \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\nu}_k(x,t_r)} \omega, \end{aligned} \quad (4.18)$$

where  $\underline{\hat{\mu}}(x, t_r) = (\hat{\mu}_1(x, t_r), \dots, \hat{\mu}_{m-1}(x, t_r))$ ,  $\underline{\hat{\nu}}(x, t_r) = (\hat{\nu}_1(x, t_r), \dots, \hat{\nu}_{m-1}(x, t_r))$ .

Let  $\theta(\underline{z}(\cdot, \cdot))$  denote the Riemann theta function ([8, 18]) on  $\mathcal{K}_{m-1}$ . Here  $\underline{z}(\cdot, \cdot)$  is defined as

$$\begin{aligned} \underline{z}(P, \underline{\hat{\mu}}(x, t_r)) &= M - \mathcal{A}(P) + \rho^{(1)}(x, t_r), \quad P \in \mathcal{K}_{m-1}, \\ \underline{z}(P, \underline{\hat{\nu}}(x, t_r)) &= M - \mathcal{A}(P) + \rho^{(2)}(x, t_r), \quad P \in \mathcal{K}_{m-1}, \end{aligned} \quad (4.19)$$

where  $M$  is the Riemann constant vector. Then the Riemann theta function representation of  $\phi(P, x, t_r)$  reads as follows.

**Theorem 4.1.** Let the curve  $\mathcal{K}_{m-1}$  be nonsingular,  $P = (\lambda, y) \in \mathcal{K}_{m-1} \setminus \{P_\infty\}$ , and  $(x, t_r), (x_0, t_{0,r}) \in \Omega_\mu$ , where  $\Omega_\mu \subseteq \mathbb{C}^2$  is open and connected. Suppose also that  $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ , or equivalently,  $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$  is nonspecial for  $(x, t_r) \in \Omega_\mu$ . Then  $\phi(P, x, t_r)$  may be explicitly constructed by the formula

$$\phi(P, x, t_r) = \frac{\theta(\underline{z}(P, \underline{\hat{\nu}}(x, t_r)))\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x, t_r)))}{\theta(\underline{z}(P_\infty, \underline{\hat{\nu}}(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x, t_r)))} \exp\left(e_\infty^{(3)}(Q_0) - \int_{Q_0}^P \omega_{P_\infty, \hat{\nu}_0(x,t_r)}^{(3)}\right). \quad (4.20)$$

**Proof.** Let  $\Phi$  denote the right hand side of (4.20), we now have to show  $\phi = \Phi$ . The Riemann theorem and (4.17) allow us to conclude that  $\Phi$  exactly has zeros at the points  $\hat{\nu}_0(x, t_r), \hat{\nu}_1(x, t_r), \dots, \hat{\nu}_{m-1}(x, t_r)$  and poles at  $P_\infty, \hat{\mu}_1(x, t_r), \dots, \hat{\mu}_{m-1}(x, t_r)$ . It infers from (4.17) that

$$\Phi \underset{\zeta \rightarrow 0}{=} \zeta^{-1}[1 + O(\zeta)], \quad P \rightarrow P_\infty, \quad (4.21)$$

which together with (4.1) gives

$$\frac{\Phi}{\phi} \underset{\zeta \rightarrow 0}{=} \frac{\zeta^{-1}[1+O(\zeta)]}{\zeta^{-1}[1+O(\zeta)]} = 1 + O(\zeta), \quad P \rightarrow P_\infty. \quad (4.22)$$

Applying the Riemann-Roch theorem, we get  $\frac{\Phi}{\phi} = 1$ , which completes the proof.

Based on the above results, we will obtain the Riemann theta function representations of solutions for the entire Dym-type hierarchy immediately.

**Theorem 4.2.** Assume that the curve  $\mathcal{K}_{m-1}$  is nonsingular and let  $(x, t_r) \in \Omega_\mu$ , where  $\Omega_\mu \subseteq \mathbb{C}^2$  is open and connected. Suppose also that  $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ , or equivalently,  $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$

is nonspecial for  $(x, t_r) \in \Omega_\mu$ . Then the Dym-type hierarchy admits algebro-geometric solutions

$$(v^{-\frac{1}{3}})_x = \partial_{U_2^{(2)}} \ln \frac{\theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r)))}{\theta(\underline{z}(P_\infty, \hat{\underline{\nu}}(x, t_r)))} + \delta(m), \quad (4.23)$$

with  $\delta(m)$  defined in (4.16).

**Proof.** From (4.9), (4.12), (4.18) and (4.19) it follows that

$$\begin{aligned} \underline{z}(P, \hat{\underline{\mu}}(x, t_r)) &= M - \int_{Q_0}^P \omega + \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\mu}_k(x, t_r)} \omega \\ &= (\dots, M_j - \int_{Q_0}^{P_\infty} \omega_j + \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\mu}_k(x, t_r)} \omega_j - \int_{P_\infty}^P \omega_j, \dots) \\ &\stackrel{\zeta \rightarrow 0}{=} (\dots, M_j - \int_{Q_0}^{P_\infty} \omega_j + \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\mu}_k(x, t_r)} \omega_j - U_{2,j}^{(2)} \zeta + O(\zeta^2), \dots), \quad P \rightarrow P_\infty. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\theta(\underline{z}(P, \hat{\underline{\mu}}(x, t_r)))}{\theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r)))} &\stackrel{\zeta \rightarrow 0}{=} \frac{\theta(\dots, M_j - \int_{Q_0}^{P_\infty} \omega_j + \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\mu}_k(x, t_r)} \omega_j - U_{2,j}^{(2)} \zeta + O(\zeta^2), \dots)}{\theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r)))} \\ &\stackrel{\zeta \rightarrow 0}{=} \frac{\theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r))) - \left[ \sum_{j=1}^{m-1} U_{2,j}^{(2)} \frac{\partial}{\partial z_j} \theta(M - \mathcal{A}(P_\infty) + \rho^{(1)}(x, t_r) - U_2^{(2)} \zeta + O(\zeta^2)) \right] \Big|_{\zeta=0}}{\theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r)))} \zeta + O(\zeta^2) \\ &\stackrel{\zeta \rightarrow 0}{=} 1 - \frac{\partial_{U_2^{(2)}} \theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r)))}{\theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r)))} \zeta + O(\zeta^2) \\ &\stackrel{\zeta \rightarrow 0}{=} 1 - [\partial_{U_2^{(2)}} \ln \theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r)))] \zeta + O(\zeta^2), \quad P \rightarrow P_\infty, \end{aligned} \quad (4.24)$$

where  $\partial_{U_2^{(2)}} \theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r))) = [\partial_{U_2^{(2)}} \theta(M - \mathcal{A}(P_\infty) + \rho^{(1)}(x, t_r) - U_2^{(2)} \zeta + O(\zeta^2))] \Big|_{\zeta=0}$  and

$\partial_{U_2^{(2)}} = \sum_{j=1}^{m-1} U_{2,j}^{(2)} \frac{\partial}{\partial z_j}$ . Quite similarly, we get

$$\frac{\theta(\underline{z}(P, \hat{\underline{\nu}}(x, t_r)))}{\theta(\underline{z}(P_\infty, \hat{\underline{\nu}}(x, t_r)))} \stackrel{\zeta \rightarrow 0}{=} 1 - [\partial_{U_2^{(2)}} \ln \theta(\underline{z}(P_\infty, \hat{\underline{\nu}}(x, t_r)))] \zeta + O(\zeta^2), \quad P \rightarrow P_\infty, \quad (4.25)$$

with  $\partial_{U_2^{(2)}} \theta(\underline{z}(P_\infty, \hat{\underline{\nu}}(x, t_r))) = [\partial_{U_2^{(2)}} \theta(M - \mathcal{A}(P_\infty) + \rho^{(2)}(x, t_r) - U_2^{(2)} \zeta + O(\zeta^2))] \Big|_{\zeta=0}$ .

By virtue of (4.17), (4.20), (4.24) and (4.25), we have

$$\begin{aligned} \phi(P, x, t_r) &\stackrel{\zeta \rightarrow 0}{=} \{1 - [\partial_{U_2^{(2)}} \ln \theta(\underline{z}(P_\infty, \hat{\underline{\nu}}(x, t_r)))] \zeta + O(\zeta^2)\} \{1 + [\partial_{U_2^{(2)}} \ln \theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r)))] \zeta \\ &\quad + O(\zeta^2)\} \times [\zeta^{-1} + \delta(m) + O(\zeta)] \\ &\stackrel{\zeta \rightarrow 0}{=} [\zeta^{-1} + \partial_{U_2^{(2)}} \ln \frac{\theta(\underline{z}(P_\infty, \hat{\underline{\mu}}(x, t_r)))}{\theta(\underline{z}(P_\infty, \hat{\underline{\nu}}(x, t_r)))} + \delta(m) + O(\zeta)], \quad P \rightarrow P_\infty. \end{aligned} \quad (4.26)$$

Comparing (4.1) with (4.26), we arrive at (4.23).

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